PAPER

Combinatorial Resonances in Coupled Duffing's Circuits

Yue MA[†a)], ***Student Member and*** **Hiroshi KAWAKAMI**[†b)], ***Regular Member***

SUMMARY In this paper, we study the fundamental combinatorial nonlinear resonances of a system consisting of two identical periodic forced circuits coupled by a linear resistor. The circuit equations are described by a system of coupled Duffing's equations. We discuss two cases of external periodic force, i.e., in-phase and anti-phase, and obtain the bifurcation diagram of each case. Periodic solutions are classified according to the symmetrical property of the circuit. Resonances in the coupled system are explained from the combinatorial standpoint. That is, we introduce the definition of combinatorial resonances and investigate the patterns of combinatorial solutions in this system.
key words: *coupled Duffing's circuits, bifurcation, combinatorial pattern*

1. Introduction

A nonlinear resonance is one of the typical phenomena in nonlinear circuits with a periodic external force. A circuit containing a saturable core [1] described by a Duffing's equation can exhibit a typical nonlinear resonance. When we consider a system of two periodic forced circuits coupled by a linear resistor, we may observe a combined resonant phenomenon, say combinatorial resonances, under weak coupling condition [2].

The system of coupled oscillators has attracted a great deal of attention due to its crucial role in understanding quasiperiodicity, transition to chaos, and the phenomena of synchronization in physical, chemical, biological and electronical systems. Especially in the last decades, there has been a great promotion in coupled oscillator theory. Mathematical basis of coupled oscillators has been established by de Figueiredo [5] and Aggawal [6], and Linkens first investigated the mutual synchronization of a large number of oscillators [7], [8]. Research of T. Endo investigated the structure and stability of various oscillatory modes of a coupled van der Pol oscillator network in ring or ladder [9], [10]. Bifurcations of equilibrium points and periodic solutions observed in coupled BVP oscillators were studied by H. Kitajima [11]. Most of the researches so far focused on the system of autonomous oscillators. But for many practical models, such as in power system, oscillating elements are described by nonautonomous systems. Because of the periodic external force, phenomena of systems coupled by nonautonomous oscillators are quite different from autonomous ones. Thus it is necessary to investigate the bifurcation of combinatorial resonances in coupled Duffing's circuit which is a typical model.

In this paper, we discuss the symmetrical property of coupled Duffing's system firstly. Based on the symmetry, we classify the external force into two cases, i.e., in-phase and anti-phase. They can exhibit different combinatorial resonances. Then the definition of combinatorial resonances is introduced, and nine combinatorial patterns are concluded. We obtain and analyze the bifurcation diagrams of these oscillations. The difference of bifurcations between these two cases is clarified [3]. Observing phase portraits of periodic solutions, we can draw some schematic amplitude characteristic diagrams of oscillations qualitatively, by which we can comprehend combinatorial resonances more intuitively and effectively. In the anti-phase case, we find out a special combinatorial resonance pattern beyond the nine patterns. In this case, the maximum number of fundamental periodic solutions increases to 11.

2. Circuit Equations

Let us consider a coupled Duffing's circuit shown in Fig. 1. With the notations in the figure, we can describe the circuit equations as follows:

$$\frac{d\phi_1}{dt} = v_1 + e_1(t)$$
$$\frac{d\phi_2}{dt} = v_2 + e_2(t)$$
$$C\frac{dv_1}{dt} + gv_1 + i_1 + G(v_1 - v_2) = 0$$

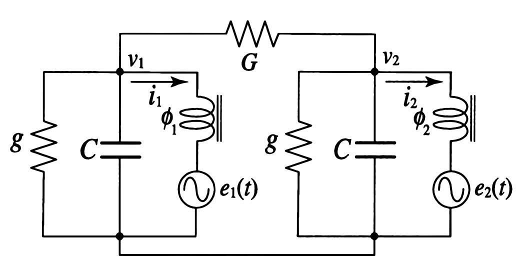

Fig. 1 A coupled Duffing's circuit.

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†The authors are with the Faculty of Engineering, The University of Tokushima, Tokushima-shi, 770-8506 Japan.
a) E-mail: mayue@ee.tokushima-u.ac.jp
b) E-mail: kawakami@ee.tokushima-u.ac.jp

$$C\frac{dv_2}{dt} + gv_2 + i_2 + G(v_2 - v_1) = 0 \tag{1}$$

where the characteristic of nonlinear inductors is assumed as following cubic functions

$$\begin{aligned} i_1 &= f(\phi_1) = f_1\phi_1 + f_3\phi_1^3 \\ i_2 &= f(\phi_2) = f_1\phi_2 + f_3\phi_2^3 \end{aligned} \tag{2}$$

Normalizing the state variables and coefficients of Eqs. (1) as

$$\begin{aligned} &\phi_1 = x_1,\ \phi_2 = x_2,\ v_1 = y_1,\ v_2 = y_2 \\ &c_1 = \frac{f_1}{C},\ c_3 = \frac{f_3}{C},\ k = \frac{g}{C},\ \delta = \frac{G}{C} \end{aligned}$$

Substitution yields

$$\begin{aligned} \frac{dx_1}{dt} &= y_1 + e_1(t) \\ \frac{dx_2}{dt} &= y_2 + e_2(t) \\ \frac{dy_1}{dt} &= -c_1x_1 - c_3x_1^3 - ky_1 - \delta(y_1 - y_2) \\ \frac{dy_2}{dt} &= -c_1x_2 - c_3x_2^3 - ky_2 - \delta(y_2 - y_1) \end{aligned} \tag{3}$$

In this paper, the external force e_1 and e_2 are discussed as two cases below:

(1) In-phase sinusoidal voltage source

$$e_1(t) = B\sin\omega t, e_2(t) = B\sin\omega t \tag{4}$$

(2) Anti-phase sinusoidal voltage source

$$e_1(t) = B\sin\omega t, e_2(t) = -B\sin\omega t \tag{5}$$

3. Symmetry, Combinatorial Resonance and Bifurcation

3.1 Symmetry of System Equations

Obviously, the system equations (3) are invariant under the coordinate transformation $(x_1, y_1) \mapsto (x_2, y_2)$ when $e_1(t) = e_2(t) = 0$. In addition, they are invariant under the inversion transformation of the signs of the variables x_1,x_2,y_1 and y_2.

To discuss the symmetry of our 4-dimensional system described as Eqs. (3), we define $\mathbf{P}$ as a matrix operation of state variable symmetrical transformation, and $\theta(t)$ as a function of time shift operation. Then the transformation operation of the system can be written as

$$\begin{aligned} g:\quad & R^4 \times R \to R^4 \times R \\ & (\mathbf{x}, t) \mapsto (\mathbf{P}\mathbf{x}, \theta(t)) \end{aligned} \tag{6}$$

Several concrete transformations are defined as follows:

$$\begin{aligned} C_2 : (\mathbf{P}, \theta(t)) &= \left(\begin{bmatrix} C & O \\ O & C \end{bmatrix}, \theta(\omega t) = \omega t\right) \\ \overline{C}_2 : (\mathbf{P}, \theta(t)) &= \left(\begin{bmatrix} \overline{C} & O \\ O & \overline{C} \end{bmatrix}, \theta(\omega t) = \omega t\right) \\ C_2\cdot\pi : (\mathbf{P}, \theta(t)) &= \left(\begin{bmatrix} C & O \\ O & C \end{bmatrix}, \theta(\omega t) = \omega t - \pi\right) \\ \overline{C}_2\cdot\pi : (\mathbf{P}, \theta(t)) &= \left(\begin{bmatrix} \overline{C} & O \\ O & \overline{C} \end{bmatrix}, \theta(\omega t) = \omega t - \pi\right) \\ \overline{I}_4\cdot\pi : (\mathbf{P}, \theta(t)) &= \left(\begin{bmatrix} \overline{I}_2 & O \\ O & \overline{I}_2 \end{bmatrix}, \theta(\omega t) = \omega t - \pi\right) \end{aligned} \tag{7}$$

where

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\overline{I}_2 = -I_2$, $\overline{C} = -C$.

We will discuss the symmetry according to the two cases stated in the previous section respectively.

(1) In-phase sinusoidal voltage source
In this case, the external force is described as Eq. (4). Observing Eqs. (3), they are invariant under the group of symmetrical operations

$$\mathbf{G} = \{I_4, C_2, \overline{C}_2\cdot\pi, \overline{I}_4\cdot\pi\} \tag{8}$$

then the product of any two of these operations is an element of $\mathbf{G}$; thus $\mathbf{G}$ is closed under the operation. The multiplication of any two elements of $\mathbf{G}$ can be represented by Table 1. This group of symmetries is called the Klein 4-group.

(2) Anti-phase sinusoidal voltage source
Similarly, in this case Eqs. (3) are invariant under the group of symmetrical operations:

$$\mathbf{G} = \{I_4, \overline{C}_2, C_2\cdot\pi, \overline{I}_4\} \tag{9}$$

3.2 Symmetry Definitions of Periodic Solutions

Consider a periodic solution of Eqs. (3)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \psi_1(t) \\ \psi_2(t) \end{bmatrix} \tag{10}$$

and its transformed solution

$$\begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \psi_1(t) \\ \psi_2(t) \end{bmatrix} \mapsto \mathbf{P}\begin{bmatrix} \varphi_1(\theta(t)) \\ \varphi_2(\theta(t)) \\ \psi_1(\theta(t)) \\ \psi_2(\theta(t)) \end{bmatrix} \tag{11}$$

Table 1 The multiplication table of $\mathbf{G}$.

	I_4	C_2	$\overline{C}_2\cdot\pi$	$\overline{I}_4\cdot\pi$
I_4	I_4	C_2	$\overline{C}_2\cdot\pi$	$\overline{I}_4\cdot\pi$
C_2	C_2	I_4	$\overline{I}_4\cdot\pi$	$\overline{C}_2\cdot\pi$
$\overline{C}_2\cdot\pi$	$\overline{C}_2\cdot\pi$	$\overline{I}_4\cdot\pi$	I_4	C_2
$\overline{I}_4\cdot\pi$	$\overline{I}_4\cdot\pi$	$\overline{C}_2\cdot\pi$	C_2	I_4

where **P** and $\theta(t)$ are defined by Eqs. (7). We define three types of solutions as follows:

(1) Completely symmetrical periodic solution

If a periodic solution is invariant under all operations in **G**, then we call this periodic solution as a completely symmetrical periodic solution. Note that the number of group orbits is only one.

(2) Inversion symmetrical periodic solution

If a periodic solution is invariant only under two operations: I_4, and $\overline{I}_4 \cdot \pi$, then the periodic solution can be called as an inversion symmetrical solution. The number of group orbits of this type of solution is two, in other words, this type of solution have a symmetric partner.

(3) Asymmetrical periodic solution

If a periodic solution is invariant only under the identity operation, then the solution is defined as an asymmetrical periodic solution. In this case, the number of group orbits is 4 which is equal to the order of the group **G**.

3.3 Combinatorial Resonances

About single Duffing's circuit, previous studies have produced completely results before decades years. It is well known that it can exhibit the nonlinear resonance with three periodic oscillations. Two of them are stable periodic oscillations corresponding to resonant and nonresonant harmonic oscillations. This phenomena can be shown in the Fig. 2. In our research, we couple two Duffing's oscillators with a linear resistor, so that these two oscillators can interact through voltage difference of this resistor. The interactions of coupled oscillators may be viewed as a complex adaptive system in which each element in the system adapts to the actions of its neighbors. Thus, many global resonance solutions generate in the case of combination. We call them combinatorial resonances.

In coupled Duffing's circuit, if we consider δ as 0, that is, the connection of two parts is broken, the system is turn to two isolated Duffing's circuits. In this special condition, combinatorial resonances of this circuit are just the completely combination of those three periodic oscillations shown in Fig. 2(b). Amount of combinatorial patterns is $3 \times 3 = 9$. All of these nine combinatorial patterns can be drawn schematically in Fig. 3. According to the symmetry definitions given in previous subsection, we can conclude that patterns P11, P22 and P33 are completely symmetrical solutions, and the others are inversion symmetrical solutions. Then, we'll discuss the bifurcations of these possible combinatorial patterns.

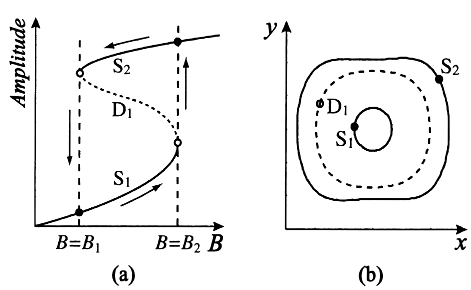

Fig. 2 Nonlinear resonance of single Duffing's circuit. (a) Amplitude characteristic diagram; (b) Phase portraits of periodic solutions.

3.4 Bifurcation of Fixed Points

By using the solutions of Eqs. (3) the Poincaré mapping T is defined as

$$T(P) = \phi(2\pi/\omega, P) \tag{12}$$

where $\phi(t, P)$ is the solution of Eqs. (3) with $\phi(0, P) = P \in R^4$. A fixed point corresponds to a periodic solution with period $2\pi/\omega$.

There are eight possible kinds of fixed points in Eqs. (3):

$$_mD\ (m = 0, 1, 2, 3, 4),\ _mI\ (m = 1, 2, 3) \tag{13}$$

The subscript m indicates the dimension of the unstable subspace. D and I denote the type of the fixed point (D is direct type and I is inverse type). When m is equal to 0 or 4, there are only D-type fixed points, we define that $_0D$ is a completely stable fixed point, and $_4D$ is a completely unstable fixed point.

A clear relation between bifurcation and every fixed point can be seen in Fig. 4.

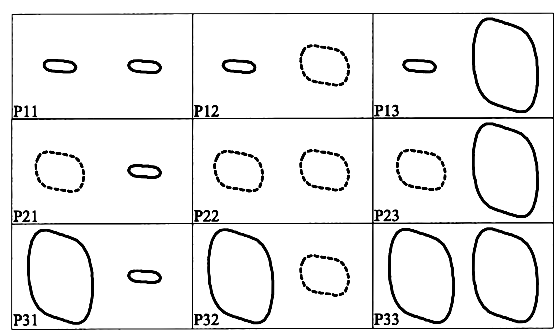

Fig. 3 Patterns of combinatorial resonances in coupled Duffing's circuit.

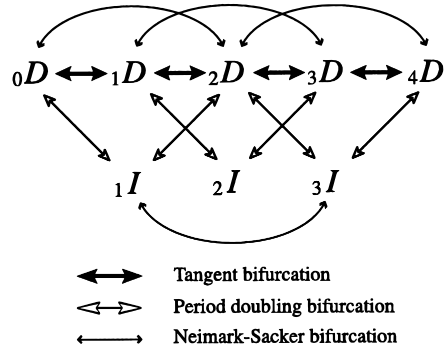

Fig. 4 The type of bifurcation and its relation with fixed points.

4. Bifurcation and Resonance

4.1 Case 1: In-Phase Sinusoidal Source

In this case, substituting Eq. (4) for Eqs. (3), we obtain the system equations as

$$\begin{aligned}\frac{dx_1}{dt} &= y_1 + B\sin(\omega t)\\ \frac{dx_2}{dt} &= y_2 + B\sin(\omega t)\\ \frac{dy_1}{dt} &= -c_1x_1 - c_3x_1^3 - ky_1 - \delta(y_1 - y_2)\\ \frac{dy_2}{dt} &= -c_1x_2 - c_3x_2^3 - ky_2 - \delta(y_2 - y_1)\end{aligned} \tag{14}$$

In Eqs. (14), we fix the system parameters as

$$c_1 = 0, c_3 = 1, k = 0.2, \omega = 1 \tag{15}$$

It is important to know the relation between combinatorial resonances and coupling intensity. So, we investigate the bifurcation problem in (B, δ) plane. The results are shown in Fig. 5. In this diagram, the notations T_n and B_n denote the tangent bifurcation and D-type of branching, respectively. The bifurcation curves separate (B, δ) plane into five regions, the number in every region indicates the number of fixed points existing in Eqs. (14), when parameters locate in each region.

A schematic diagram of amplitude characteristics of the periodic solutions of Eqs. (14) is shown in Fig. 6, when the parameter B changes along the line L ($\delta = 0.1$) in Fig. 5.

In Fig. 6 the symbol n_mD indicates the property of an amplitude curve. It means n numbers of periodic solution(s) with the type of $_mD$. We draw stable solutions with solid curves, and unstable ones with dashed curves. The thick curve which looks like an 'S' represents the completely symmetrical solution. They correspond to P33, P22 and P11 pattern from the top down respectively. The thin curve branching out from thick curve at B_1 represents inversion symmetrical solution. This type of solution always appear in pairs, but we describe them by one amplitude curve since two branches own the same amplitude and bifurcation structure. When parameter B increases, solutions patterns P23 and P32 generate with D-type of branching. Then, tangent bifurcation T_3 brings two pairs of solutions: P13, P31 and P12, P21. With the variation of resonance amplitude, P13 and P23 can merge and disappear at T_4 as well as P31 and P32. At last, P12 and P21 integrate with P22 at B_1.

We analyzed the bifurcations of combinatorial resonances in (B, δ) parameter plane above. If external force of coupled Duffing's circuit content a direct voltage component, we can describe $e(t)$ in Eqs. (3) as:

$$e_1(t) = e_2(t) = B\sin(\omega t) + B_0$$

then, we investigate the bifurcation problem in (B, B_0) plane. In this condition, although many complex subharmonic resonance appear, we only consider fundamental harmonic resonance in this paper. With the parameter δ is fixed at 0.05, the result is shown in Fig. 7. In this bifurcation diagram, notations have same meaning as those in Fig. 5, and bifurcation curves have almost same structure. Because of direct voltage component, the dash curve between a and b on T_3 turns to the tangent bifurcation of $_1I$ and $_2I$ type of solutions, instead of $_0D$ and $_1D$ type of solutions. This phenomenon was discussed in Ref. [4].

From the bifurcation diagrams Fig. 5 and Fig. 7, we may conclude that all the nine patterns coexist only when satisfy weak coupling condition, more exactely, when parameters locate in the region surrounded by T_3 and T_4. There is only one pair of stable periodic solutions with inversion symmetry, we can show its phase portrait in Fig. 8. Moreover, δ make no difference to completely symmetrical solutions, because in this case, the terms of coupling are equal to 0 when $y_1 = y_2$.

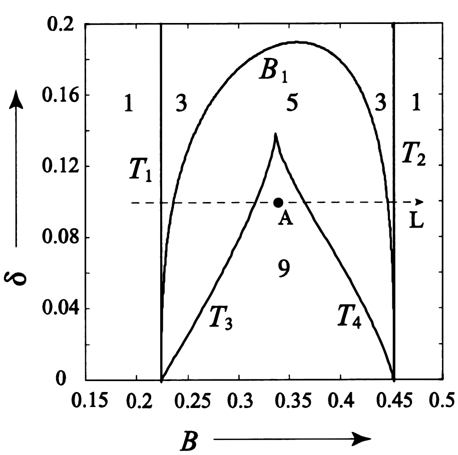

Fig. 5 Bifurcation diagram in (B, δ) plane of fixed points of in-phase case.

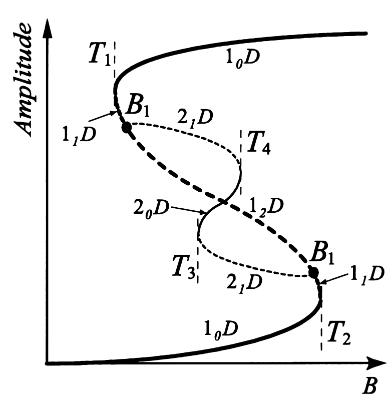

Fig. 6 An amplitude characteristic diagram corresponding to the line L in Fig. 5.

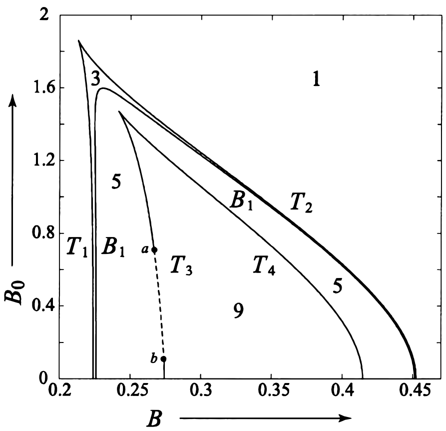

Fig. 7 Bifurcation diagram in (B, B_0) plane of fixed points of in-phase case.

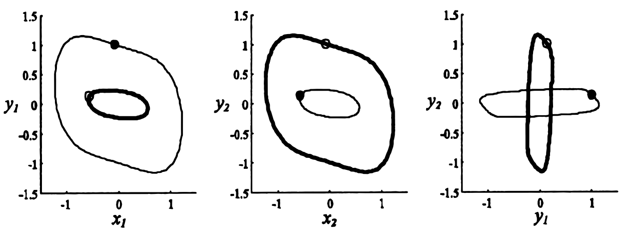

Fig. 8 Phase portraits of stable solutions when parameters locate at point A ($B = 0.34, \delta = 0.1, B_0 = 0$) in Fig. 5. We show them in (x_1, y_1), (x_2, y_2) and (y_1, y_2) plane respectively. Two solutions are described by thin and thick curve respectively; markers • and ∘ represent fixed points of these two periodic solutions.

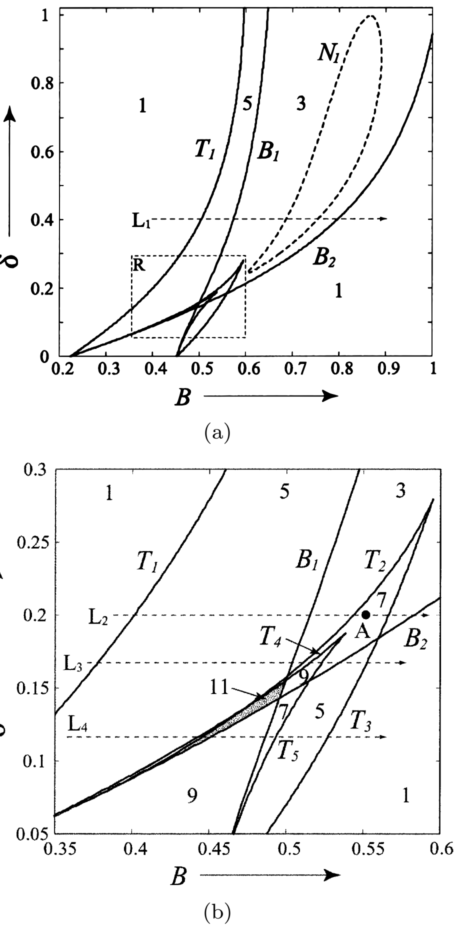

Fig. 9 (a) Bifurcation diagram in (B, δ) plane of fixed points of anti-phase case; (b) Enlarged diagram of dashed rectangle in (a).

4.2 Case 2: Anti-Phase Sinusoidal Source

In this case, we analyze with the same method as in previous case. Substituting Eq. (5) for Eqs. (3) yields:

$$\frac{dx_1}{dt} = y_1 + B\sin(\omega t)$$
$$\frac{dx_2}{dt} = y_2 - B\sin(\omega t)$$
$$\frac{dy_1}{dt} = -c_1x_1 - c_3x_1^3 - ky_1 - \delta(y_1 - y_2)$$
$$\frac{dy_2}{dt} = -c_1x_2 - c_3x_2^3 - ky_2 - \delta(y_2 - y_1) \qquad (16)$$

When $y_1 = y_2$, equations above can not keep in-phase any more, so this system exhibit stronger nonlinearity than case 1. In Eqs. (16), we also fix parameters as Eq. (15). Then we can show a bifurcation diagram in (B, δ) plane as Fig. 9(a), and details in the region surrounded by dashed rectangle R are expressed in Fig. 9(b).

In Fig. 9(a), the dashed curve N_1 indicates a Neimark-Sacker bifurcation. Other notations have the same meaning as defined in previous case. To think about the bifurcation of this case intuitively, we draw the schematic amplitude characteristic in Fig. 10 when parameters change along lines $L_{1,2,3,4}$ in Fig. 9.

Similarly to the previous case, the inversion symmetrical solutions are branch out from completely symmetrical solutions by D-type of branching. But the pair of stable inversion symmetrical solution diverge from B_2 is significant because it is a new combinatorial pattern besides nine patterns introduced in Fig. 3. It occurs because of strong reciprocity when anti-phase coupled Duffing's circuit in resonant state, and it is impossible in in-phase case. We consider it as a compromise of pattern P33 and the pair P13, P31. This combinatorial resonance is tend to occur a quasi-periodic oscillation by a Neimark-Sacker bifurcation (see Fig. 10(a)). When δ small enough to across T_2 and T_3, two pairs of stable inversion symmetrical solutions can coexist shown as Fig. 10(b). Then, completely symmetrical solution occur tangent bifurcation at T_4 and T_5 (Fig. 10(c)). Figure 10(d) shows a state that 11 periodic solutions coexist in this system after B_1 across T_4 and T_2. This special parameter area is expressed as shadow region ▒ in Fig. 9(b).

To compare these two stable inversion symmetri-

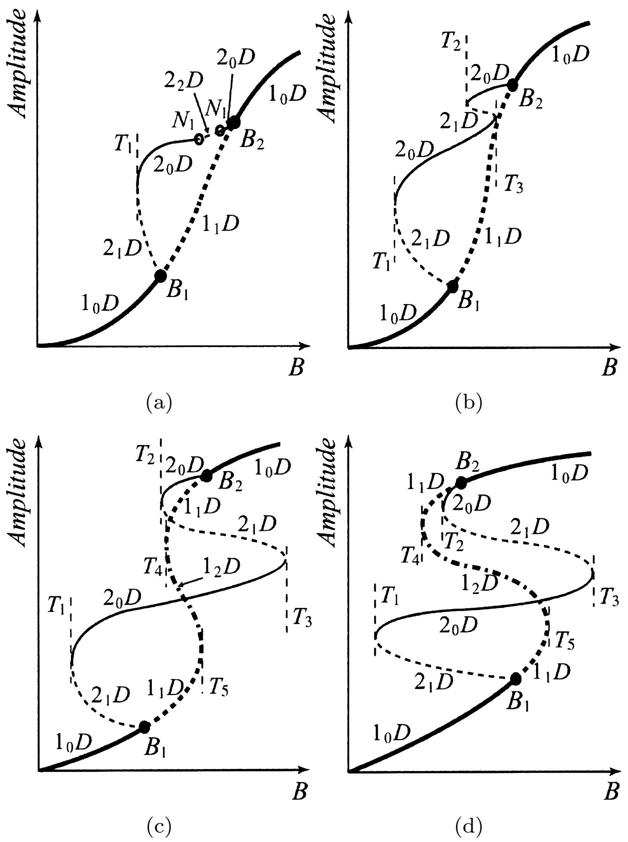

Fig. 10 Amplitude characteristic diagrams corresponding to (a) $L_1(\delta = 0.4)$; (b) $L_2(\delta = 0.2)$; (c) $L_3(\delta = 0.17)$; (d) L_4 $(\delta = 0.12)$ in Fig. 9, respectively.

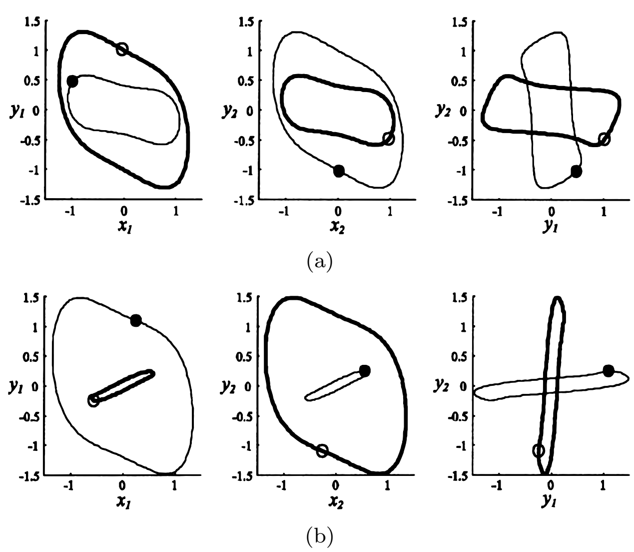

Fig. 11 Phase portraits of two pairs of stable inversion symmetrical solutions when parameters at point A$(B = 0.55, \delta = 0.2)$ on L_2 in Fig.9(b).

cal solutions, we draw their phase portraits in Fig. 11. Figure 11(a) represents the new combinatorial pattern which correspond to the upper solid inversion symmet-

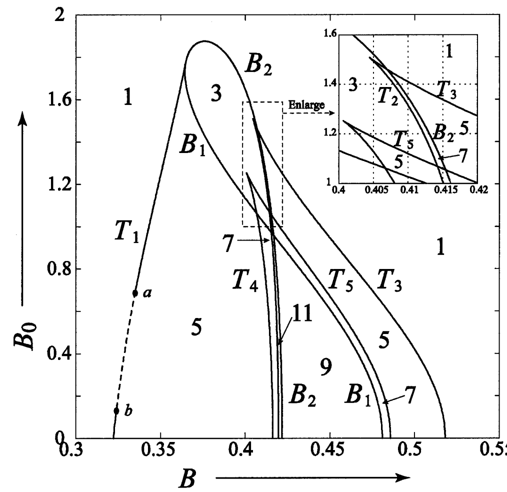

Fig. 12 Bifurcation diagram in (B, B_0) plane of fixed points of anti-phase case.

rical solution curve in Fig. 10(b); diagram (b) represents the below one.

In addition, if we add a DC voltage component in external force, Eqs. (5) can be rewrited as:

$$e_1(t) = B\sin(\omega t) + B_0, e_2(t) = -B\sin(\omega t) + B_0$$

then we investigate the bifurcation problem in (B, B_0) plane. Similarly to in-phase case, we simply consider fundamental harmonic resonance. Fixing parameter δ as 0.1, results can be shown in Fig. 12. Notations in this figure have same meaning as those in Fig. 9. The dash part between a and b is also the tangent bifurcation of ${}_1I$ and ${}_2I$ type of solutions (please refer to Ref. [4]).

From Fig. 9(b), you may note that curves T_2 and B_2 will combine together when δ is sufficient small. As to Fig. 12, T_2 will disappear and T_3 will be tangent to B_2. Hence the combinatorial resonant solutions shown in Fig. 11(a) disappear.

5. Concluding Remarks

We have investigated the combinatorial resonances in coupled Duffing's circuit. Two cases of external force were considered. We analyzed the symmetrical property, and defined symmetry patterns of periodic solutions. Then we obtained bifurcation diagrams of each case.

Comparing in-phase and anti-phase case, we conclude that two cases exhibit similar combinatorial resonances under very weak coupling condition. As a special example, when coupling intensity parameter δ is 0, both cases exhibit the resonances shown in amplitude characteristic diagram Fig. 13. In this special condition, three bifurcations degenerate to one, and system have 9 periodic solutions at most. Increasing δ, the coupled Duffing's circuits exhibit various combinatorial resonance phenomena as described in this paper.

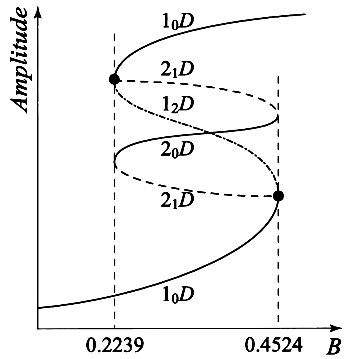

Fig. 13 Amplitude characteristic diagram of 9 periodic solutions when δ is 0.

In in-phase case, δ has no effect on the number of completely symmetrical solutions. Contrarily, variation of δ affects the number of completely symmetrical solutions and their stability in anti-phase case.

Specially, coupled Duffing's circuit excitated by anti-phase sinusoidal voltage source can occur 11 combinatorial resonance solutions. This number is beyond the combinatorial number of resonances, i.e., $3 \times 3 = 9$.

A combinatorial resonance phenomena caused by a finite number of coupled oscillators is an interesting problem in nonlinear circuit open to the future.

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Yue Ma was born in Mudanjian, China, on October 30, 1976. He received the B.E. degree from The Harbin Institute of Technology (HIT), Harbin, China, in 1998. He came to Japan in 1999 as an international exchange student and began his study of nonlinear science. He receirved M.E. from HIT in 2000. Currently, he is studying toward the Ph.D. degree at the University of Tokushima, Japan. His research interest is the bifurcation problem in nonlinear systems.

Hiroshi Kawakami was born in Tokushima, Japan, on December 6, 1941. He received the B.Eng. degree from the University of Tokushima, Tokushima, Japan, in 1964, the M.Eng. and Dr.Eng. degrees from Kyoto University, Kyoto, Japan, in 1966 and 1974, respectively, all in electrical engineering. Presently, he is a Professor of Electrical and Electronic Engineering, the University of Tokushima, Tokushima, Japan. His interest is qualitative properties of nonlinear circuits.