



Control of chaos in a piecewise smooth nonlinear system

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Abstract

This paper shows the stabilization of the unstable periodic orbit of any given piecewise smooth system with linear and/or nonlinear characteristics. By utilizing the periodicity of the switching action, we construct the Poincaré mapping including all information of the original system. This mapping offers a first step toward extending a novel technique for controlling chaos based on the appropriate state feedback in piecewise smooth nonlinear systems. We also apply this approach to Rayleigh type oscillator described by the piecewise smooth nonlinear systems.

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1. Introduction

There have been various investigations on controlling chaos in the piecewise systems. For the piecewise linear systems, Saito and Mitsubori proposed a control method of the piecewise linear version for the occasional proportional feedback method [1]. This method showed theoretical proof of chaos generation and identification of the unstable periodic orbit. Theoretical result was also verified by laboratory experiments. On the other hand, many interesting bifurcation phenomena are observed for several power electronic systems, both numerically and experimentally [2–6]. In addition, just as techniques have been developed over the years to control the concrete systems, for example, the current or voltage-mode controlled converters, PWM current-mode H-bridge inverter, and so on [7–12]. However, if the dynamics of the system describes the nonlinear ordinary differential equations, it is impossible to calculate the exact solution. Moreover, the continuous systems corresponding to the piecewise smooth system have interrupted characteristics. Therefore, we must obtain the information of the unstable periodic points and their multipliers by using an appropriate numerical integration. This directly implies that the conventional methods do not give us a reliable and correct information.

The present paper shows a method to stabilize the chaotic attractor observed in piecewise smooth nonlinear systems. In [13], we have developed a simple hybrid system exhibiting chaos, chaos control, and its circuit realization. This paper offers a first step toward extending the general control method of chaos in the piecewise smooth nonlinear systems. Firstly, we define two mappings by using the periodicity of the switching action. This assures us to obtain the

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information of the unstable periodic orbit. By using above information, the appropriate state feedback method for the corresponding discrete system derived from such a piecewise smooth system via composite Poincaré mapping works effectively. As an illustrate example, we stabilize the unstable period- r orbit in two dimensional piecewise nonlinear system.

2. Control scheme

2.1. Description of problem

Let us consider two n -dimensional autonomous systems,

$$\text{system a : } \frac{dx}{dt} = f_a(x, \lambda, \lambda_a), \tag{1}$$

$$\text{system b : } \frac{dx}{dt} = f_b(x, \lambda, \lambda_b). \tag{2}$$

We assume that the orbit of the systems is described by

$$x(t) = \varphi(t, x_0, \lambda, \lambda_a, \lambda_b), \tag{3}$$

where, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}^r$ are common parameters for f_a and f_b . $\lambda_a, \lambda_b \in \mathbf{R}^s$ are parameters depending only on f_a and f_b , respectively. Given that f_a and f_b are C^∞ -class maps for any variables and parameters. Note that, (8) is the mixed orbit of (1) and (2). The behavior of the orbit is shown in Fig. 1. Next, let, x^* be an unstable fixed or periodic point in (1) and (2). $\lambda^* \in \mathbf{R}^r$, $\lambda_a^* \in \mathbf{R}^s$ and $\lambda_b^* \in \mathbf{R}^s$ are parameters giving the fixed point x^* . Assume that is λ_a controlling parameter. We explicitly define the local section Π by using one scalar function $q: \mathbf{R}^n \rightarrow \mathbf{R}$:

$$\Pi = \{x \in \mathbf{R}^n | q(x) = 0\}. \tag{4}$$

Suppose that the orbit starts with (1), and then changes to system b defined by (2) if it intersects a local section (4) at time τ_a . Any clock pulse before the intersection is ignored. Then the orbit remains as (2) until the next clock cycle begins. The time interval of (2) is

$$\tau_b = T \left[1 - \left(\frac{\tau_a}{T} \bmod 1 \right) \right]. \tag{5}$$

After that, the orbit obeys (1) again. The behavior of the orbit is sketched schematically as Fig. 1.

2.2. Design of a controller

In the following, we consider a method for stabilizing the unstable period- m orbit in the chaotic attractor. We define the following two mappings:

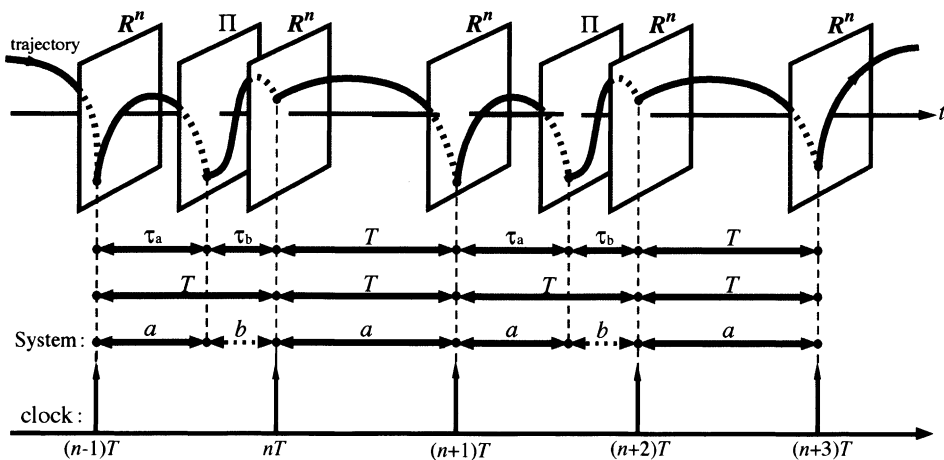


Fig. 1. Behavior of the orbit and Poincaré map M .

(1) In case that the orbit governed by (1) and starting from \mathbf{x}_k intersects the local section (4) beyond time $T(\tau_a > T)$:

$$M_k : \mathbf{R}^n \rightarrow \mathbf{R}^n, \quad \mathbf{x}_k \mapsto \mathbf{x}_{k+1} = \varphi_a(T, \mathbf{x}_k, \lambda^*, \lambda_a^*). \tag{6}$$

The variations of (1) for the neighborhood of the unstable periodic point \mathbf{x}_k is

$$\mathbf{x}(k) = \mathbf{x}_k^* + \xi(k), \quad \lambda(k) = \lambda_a^* + \mathbf{u}(k). \tag{7}$$

From (1) and (7), the variational equation of the unstable periodic point is described by

$$\xi = (k+1) = \frac{\partial M_k}{\partial \mathbf{x}_k^*} \xi(k) + \frac{\partial M_k}{\partial \lambda_a^*} \mathbf{u}(k) = \widehat{A}_k \xi(k) + \widehat{B}_k \mathbf{u}(k). \tag{8}$$

(2) On the other hand, if $\tau_a \leq T$, we construct the following Poincaré mapping:

$$\begin{aligned} M_a : \mathbf{R}^n &\rightarrow \Pi \\ \mathbf{x}_k^* &\mapsto \mathbf{x}_q^* = \varphi_a(\tau_a(\mathbf{x}_k^*), \mathbf{x}_k^*, \lambda^*, \lambda_a^*), \\ M_b : \Pi &\rightarrow \mathbf{R}^n \\ \mathbf{x}_q^* &\mapsto \mathbf{x}_{k+1}^* = \varphi_b(\tau_b, \mathbf{x}_q^*, \lambda^*, \lambda_b^*), \\ M_k : \mathbf{R}^n &\rightarrow \mathbf{R}^n \\ \mathbf{x}_k^* &\mapsto \mathbf{x}_{k+1}^* = M_b \circ M_a. \end{aligned} \tag{9}$$

As a result, the variational equation as follows is described by

$$\xi(k+1) = \frac{\partial M_b}{\partial \mathbf{x}_q^*} \frac{\partial M_a}{\partial \mathbf{x}_k^*} \xi(k) + \frac{\partial M_b}{\partial \mathbf{x}_q^*} \frac{\partial M_a}{\partial \lambda_a^*} \mathbf{u}(k) = \widetilde{A}_k \xi(k) + \widetilde{B}_k \mathbf{u}(k), \tag{10}$$

where

$$\begin{aligned} \frac{\partial M_a}{\partial \mathbf{x}_k^*} &= \left[\mathbf{I}_n - \frac{1}{\frac{\partial q}{\partial \mathbf{x}} \cdot \mathbf{f}_a} \mathbf{f}_a \cdot \frac{\partial q_1}{\partial \mathbf{x}} \right] \frac{\partial \varphi_a}{\partial \mathbf{x}_k}, \\ \frac{\partial M_a}{\partial \lambda_k^*} &= \left[\mathbf{I}_n - \frac{1}{\frac{\partial q}{\partial \mathbf{x}} \cdot \mathbf{f}_a} \mathbf{f}_a \cdot \frac{\partial q_1}{\partial \mathbf{x}} \right] \frac{\partial \varphi_a}{\partial \lambda_a}, \\ \frac{\partial M_b}{\partial \mathbf{x}_q^*} &= \frac{\partial \varphi_b}{\partial \mathbf{x}_q^*}. \end{aligned} \tag{11}$$

In order to calculate $\partial \varphi_k / \partial \mathbf{x}_k$, an appropriate numerical integration is applied for the following differential equation:

$$\frac{d}{dt} \left(\frac{\partial \varphi_k}{\partial \mathbf{x}_k} \right) = \frac{\partial \mathbf{f}_k}{\partial \mathbf{x}} \left(\frac{\partial \varphi_k}{\partial \mathbf{x}_k} \right), \quad \left. \frac{\partial \varphi_k}{\partial \mathbf{x}_k} \right|_{t=0} = \mathbf{I}_n. \tag{12}$$

By using (6) and (9), we define a composite Poincaré map:

$$M = M_{m-1} \circ M_{m-2} \circ \dots \circ M_0. \tag{13}$$

From (8) and (10), the variational equation, of period- m fixed point is obtained as

$$\xi(k+m) = \mathbf{A} \xi(k) + \mathbf{B} \mathbf{u}(k). \tag{14}$$

where \mathbf{A} and \mathbf{B} are equal to the product M_k at each map. Either (8) or (10) is used in the variational equation of M_i . Finally, we construct the state feedback for (14):

$$\mathbf{u} = \mathbf{C}^T \xi = \mathbf{C}^T (\mathbf{x} - \mathbf{x}_0^*), \tag{15}$$

when \mathbf{C} is a control matrix ($r \times n$). Then, characteristic, equation of controlled system is

$$|\mathbf{A} + \mathbf{B} \mathbf{C}^T - \mu \mathbf{I}_n| = 0. \tag{16}$$

As a result, any controller based on the state feedback can be applied.

3. An illustrative example

We shall provide numerical example to verify our theoretical result. Fig. 2 shows the Rayleigh type oscillator containing a state-period dependent switch. Note that, this circuit has a nonlinear conductor, which makes it impossible to obtain the fixed or periodic points by using the exact solution. After the relabeling

$$g_1 = \frac{1}{R_0 + R_1}, \quad g_2 = \frac{1}{R_0 + R_2}, \quad B_1 = g_1 E_1, \quad B_2 = g_2 E_2, \tag{17}$$

the circuit equations become

$$\text{SW : a} \begin{cases} \frac{dx}{dt} = f_x(x, y, \lambda, B_1) = -rx - y, \\ \frac{dy}{dt} = f_y(x, y, \lambda, B_1) \\ = x + (1 - g_1)y - \frac{1}{3}y^3 + B_1, \end{cases} \tag{18}$$

$$\text{SW : b} \begin{cases} \frac{dx}{dt} = g_x(x, y, \lambda, B_2) = -rx - y, \\ \frac{dy}{dt} = g_y(x, y, \lambda, B_2) \\ = x + (1 - g_2)y - \frac{1}{3}y^3 + B_2, \end{cases} \tag{19}$$

where the characteristics of the nonlinear conductance is assumed as

$$G(y) = -y + \frac{1}{3}y^3. \tag{20}$$

We fix the following parameters $r = 0.1, g_1 = 0.2, g_2 = 2.0, B_1 = 1.206, B_2 = 2.0, y_{\text{ref}} = -0.91, T = 5.0$. Under these conditions, we can observe the chaotic attractor shown in Fig. 3 via period doubling bifurcation and border-collision bifurcation [14]. The scalar function q can be described by

$$\Pi = \{(x, y) \in \mathbf{R}^2 | q(y) = y_{\text{ref}}\}. \tag{21}$$

In the following, we choose B_1 as the control parameter and briefly explain the control method of the unstable period-4 orbit in the chaotic attractor. By utilizing the periodicity of the switching action, we first construct the local mapping, and then the Poincaré mapping can be constructed as a composite map of local mappings. If the orbit starting at (x_k^*, y_k^*) with (18) intersects the local section beyond time $T(\tau_a > T)$, the local mapping is given by

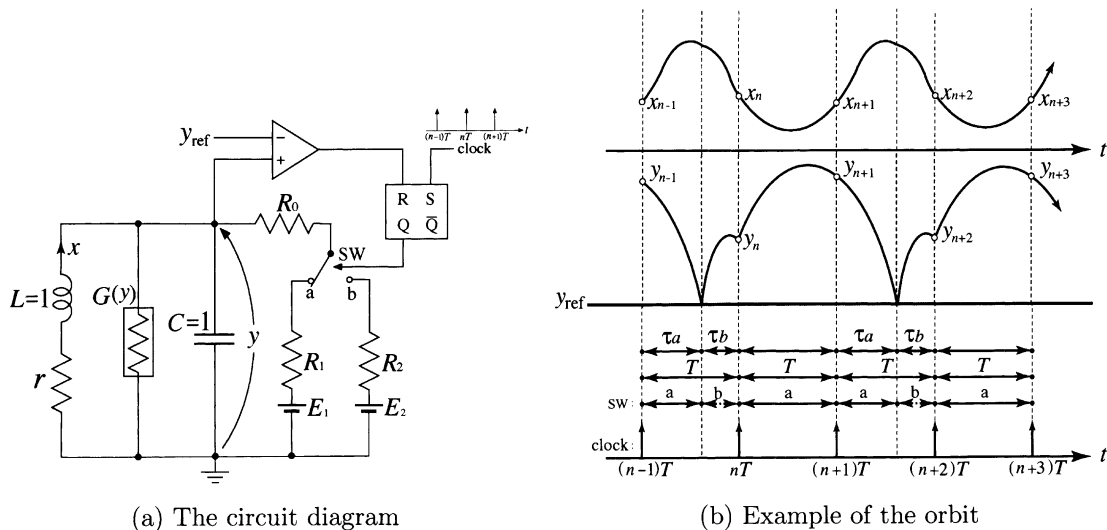


Fig. 2. Rayleigh type oscillator containing a state-period dependent switch.

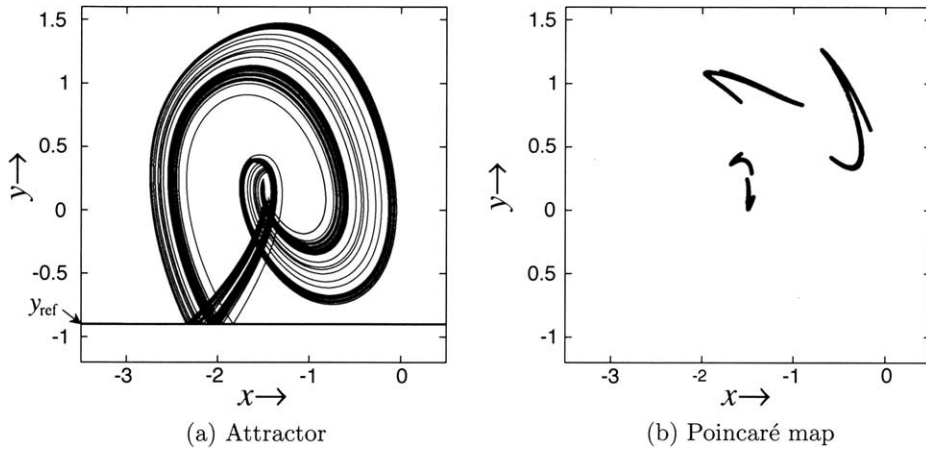


Fig. 3. The chaotic attractor.

$$\begin{aligned}
 M_k : \mathbf{R}^2 &\rightarrow \mathbf{R}^2, \\
 x_k^* &\rightarrow x_{k+1}^* = \varphi_x(T, x_k^*, y_k^*, \lambda^*, B_1^*), \\
 y_k^* &\rightarrow y_{k+1}^* = \varphi_y(T, x_k^*, y_k^*, \lambda^*, B_1^*),
 \end{aligned} \tag{22}$$

where the variations of x_k^* and y_k^* are

$$x(k) = x_k^* + \zeta(k), \quad y(k) = y_k^* + \eta(k), \quad \lambda(k) = B_1^* + \mathbf{u}(k). \tag{23}$$

The variational equation is

$$\zeta(k+1) = \widehat{\mathbf{A}}_k \zeta(k) + \widehat{\mathbf{B}}_k \mathbf{u}(k), \tag{24}$$

where

$$\begin{aligned}
 \widehat{\mathbf{A}}_k &= \begin{bmatrix} \frac{\partial \varphi_x}{\partial x_k^*} & \frac{\partial \varphi_x}{\partial y_k^*} \\ \frac{\partial \varphi_y}{\partial x_k^*} & \frac{\partial \varphi_y}{\partial y_k^*} \end{bmatrix}, \quad \widehat{\mathbf{B}}_k = \begin{bmatrix} \frac{\partial \varphi_x}{\partial B_1^*} \\ \frac{\partial \varphi_y}{\partial B_1^*} \end{bmatrix}, \\
 \zeta(k) &= \begin{bmatrix} \zeta(k) \\ \eta(k) \end{bmatrix}, \quad \mathbf{u}(k) = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \zeta(k).
 \end{aligned} \tag{25}$$

On the other hand, if $\tau_a \leq T$, we construct the following Poincaré mapping:

$$\begin{aligned}
 M_a : \mathbf{R}^2 &\rightarrow \Pi, \\
 x_k^* &\rightarrow x_q^* = \varphi_x(\tau_a, x_k^*, y_k^*, x_k^*, y_k^*, \lambda^*, B_1^*), \\
 y_k^* &\rightarrow y_q^* = y_{\text{ref}}, \\
 M_b : \Pi &\rightarrow \mathbf{R}^2, \\
 x_q^* &\rightarrow x_{k+1}^* = \phi_x(\tau_b, x_q^*, y_{\text{ref}}^*, \lambda^*, B_2^*), \\
 y_{\text{ref}}^* &\rightarrow y_{k+1}^* = \phi_y(\tau_b, x_q^*, y_{\text{ref}}^*, \lambda^*, B_2^*), \\
 M_k : \mathbf{R}^2 &\rightarrow \mathbf{R}^2, \\
 x_k^* &\rightarrow x_{k+1}^*, \\
 y_0^* &\rightarrow y_{k+1}^* = M_b \circ M_a.
 \end{aligned} \tag{26}$$

The variational equation is described by

$$\zeta(k+1) = \widetilde{\mathbf{A}}_k \zeta(k) + \widetilde{\mathbf{B}}_k \mathbf{u}(k), \tag{27}$$

where

$$\hat{A}_k = \begin{bmatrix} \frac{\partial \phi_x}{\partial x_q^*} & \frac{\partial \phi_x}{\partial y_{\text{ref}}^*} \\ \frac{\partial \phi_y}{\partial x_q^*} & \frac{\partial \phi_y}{\partial y_{\text{ref}}^*} \end{bmatrix} \begin{bmatrix} X_k \frac{\partial \varphi_x}{\partial x_k^*} & Y_k \frac{\partial \varphi_x}{\partial y_k^*} \\ X_k \frac{\partial v_{\text{ref}}}{\partial x_k^*} & Y_k \frac{\partial v_{\text{ref}}}{\partial y_k^*} \end{bmatrix},$$

$$\hat{B}_k = \begin{bmatrix} \frac{\partial \phi_x}{\partial x_q^*} & \frac{\partial \phi_x}{\partial y_{\text{ref}}^*} \\ \frac{\partial \phi_y}{\partial x_q^*} & \frac{\partial \phi_y}{\partial y_{\text{ref}}^*} \end{bmatrix} \begin{bmatrix} X_k \frac{\partial \varphi_x}{\partial B_1^*} \\ Y_k \frac{\partial v_{\text{ref}}}{\partial B_1^*} \end{bmatrix}, \tag{28}$$

$$X_k = \left[\mathbf{I}_2 - \frac{1}{\frac{\partial q}{\partial x_k^*} \cdot f_x} f_x \cdot \frac{\partial q}{\partial x_k^*} \right], \quad Y_k = \left[\mathbf{I}_2 - \frac{1}{\frac{\partial q}{\partial y_k^*} \cdot f_y} f_y \cdot \frac{\partial q}{\partial y_k^*} \right].$$

As a result, the composite Poincaré map of the unstable period-4 fixed point is given by

$$M = M_3 \circ M_2 \circ M_1 \circ M_0. \tag{29}$$

Furthermore, the variational equation of period-4 fixed point is

$$\zeta(k+4) = A\zeta(k) + Bu(k). \tag{30}$$

The derivative of the Poincaré map is equal to the product of the derivative of each local maps. With above parameters, the location (x,y) of the unstable period-4 fixed point is $(-1.488, 0.123)$, and its multipliers are $(-0.126, -1.889)$, respectively. By using the following state feedback $u(k)$:

$$u(k) = G_1(i_k - i^*) + G_2(v_k - v^*), \tag{31}$$

the unstable period-4 orbit in the chaotic attractor can be stabilized due to the linear control technique with stable pole assignment. We now place the poles to realized dead-beat control. From this, we can obtain the control gain $G_1 = -0.515$ and $G_2 = -0.619$. Fig. 4 shows the numerical simulation of the stabilized period-4 orbit and its Poincaré map.

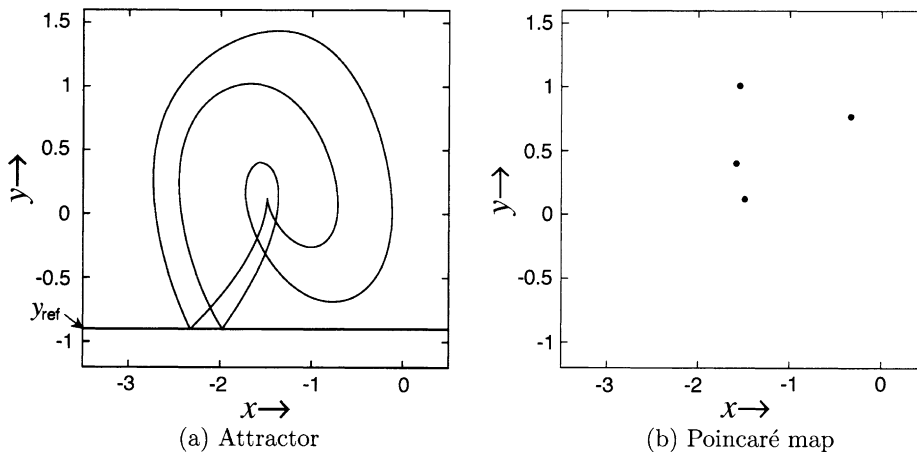


Fig. 4. Stabilized period-4 orbit.

4. Conclusions

This paper has proposed a control method of chaos in piecewise smooth nonlinear systems. Deriving discrete mapping as Poincaré mapping and the controller were designed with the feedback method by using the calculation technique for numerical integration. This method can be used directly in the application of engineering principles in the current or voltage-mode controlled converters. Furthermore, we stabilized the unstable period-4, orbit of the chaotic attractor in Rayleigh type oscillator containing a state-period dependent switch.

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